Operators of Approximation of Functions f(x, y) by their Projections on the System of Nonparallel Lines for Computed Tomography

Vitaliy Mezhuyev, Oleh M. Lytvyn, Oleh O. Lytvyn, Iuliia I. Pershyna, Olesia P. Nechuiviter, and Yevheniia L. Khurdei

***Abstract*—Operators of approximation of the functions of two variables, interpolating the functions by their projections along M nonparallel lines, were not sufficiently considered in the scientific literature. At the same time, this theoretical problem has a strong practical interest when the given projections (integrals along lines) come from a computed tomography scanner. The paper constructs the interpolation operator which exactly restores the polynomials of degree M-1. The method was investigated for a system of mutually perpendicular lines and for three nonparallel intersecting lines (sides of a triangle). An integral representation of the residual member of the approximation by the obtained operators for differentiable functions is found. The proposed method allows to expand theory and practical applications of computed tomography.**

***Index Terms*—Computed tomography, approximation, operators of interpolation, projections among lines.**

I. Introduction

Modern science and technology (astronomy, radiology, computed tomography, holography, radar technology etc.) based on the use of digital tools, algorithms and methods. Mentioned domains need a development of mathematical models and methods, applicable to the new types of input information. An important case is when an input information is given as the set of traces of a function on the planes or on the lines. The theory of interlineation and interflatation of functions gives the general operators to solve such types of problems. Papers [1], [2] demonstrated, how the operators of interlineation and interflatation of functions can be used to develop a method of numerical integration of highly oscillating functions of many variables.

Works [3]-[5] constructed the new information operators to solve the problems of computed tomography, where the paper [5] considers the heterogeneity of an internal structure of the body.

Works [6]-[8] devoted to solving the problems of computed tomography based on the use of direct and inverse Radon transform, Fourier direct method etc.

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Vitaliy Mezhuyev is with the University Malaysia Pahang, Gambang, Malaysia (e-mail: vitaliy@ump.edu.my).

Oleh M. Lytvyn, Oleh O. Lytvyn, Iuliia I. Pershyna, Olesia P. Nechuiviter and Yevheniia L. Khurdei are with the Ukrainian Engineering and Pedagogical Academy, Kharkiv, Ukraine (e-mails: academ_mail@ukr.net, loo71@bk.ru, yulia_pershina@mail.ru, olesya@email.com, evgesha.hurdei@gmail.com).

Works [9] and [10] proposed a new approach for constructing of operators of approximation of functions of two variables by the use of known projections along a given system of lines. Given approach also takes into account the points of intersection of M lines.

Work [10] investigated the case when a system of lines is mutually perpendicular. Works [11] and [12] considered the cases of three non-parallel crossed lines and the system of M crossed lines, which do not intersect at one point. Work [13] investigated the method of approximation of functions of two variables, where the projections along the sides of a triangle were used an input information.

For the best or our knowledge, there is no existing method for constructing operators for the approximation of the functions of two variables, which uses the interpolation data given on the system of non-parallel lines and also projections – integrals along the given system of lines.

It should be noted that within the framework of classical computed tomography only the projections (integrals) are used. In the works [4], [5] the stated problem is formulated, but an explicit analytic solution is given only for the case when the systems of lines are mutually perpendicular.

This paper constructs the interpolation operators which use the known projections of functions along given M nonparallel lines, which have no more than one intersection at one point. The novelty of the proposed approach is the selection of unknown interpolation values of the approximate function at the points of intersection of the lines. For it the operator, built by interpolation data at the points of intersection of given lines, must accurately restore the polynomials of *M*-1 (*M*=3) degree or it is found from the condition of a minimum of the operator by the method of least squares.

II. Constructing of the Operators for Interpolation of Functions $f(x,y)$ at the Points of Intersection of M Lines

$$f(x,y)\in C(D),\ D=\mathrm{supp}(D)$$

Let us given the area $D\subset R^2, diam<\infty$, $\Gamma_k^*=D\cap\Gamma_k\neq\theta, k=\overline{1,M}$. Let the function $f(x,y)$ has the following properties: $f(x,y)\in C(D)$, $f(x,y)=\mathrm{supp}(D)$. Let's consider M nonparallel lines, which are described by the equations $\Gamma_k:\omega_k(x,y)\equiv\alpha_k\cdot x+\beta_k\cdot y-\varphi_k=0$, $k=\overline{1,M}$, $a_k^2+b_k^2=1$.

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Let's build the operator $L1_M f(x,y)$, which interpolates $f(x,y)$ in the points $(X_{pq},Y_{pq})\in\Gamma_p\cap\Gamma_q$, $p,q=\overline{1,M}$, $p\neq q$ as:

$$L1_M f(x,y)=\sum_{k=1}^{M}\sum_{\substack{l=1\\ l\neq k\\ l<k}}^{M} f(X_{kl},Y_{kl})h_{kl}(x,y), \qquad (1)$$

where $h_{kl}(x,y)=\prod_{\substack{j=1\\ j\neq k\\ j\neq l}}^{M}\frac{\omega_j(x,y)}{\omega_j(X_{kl},Y_{kl})}$.

Lemma 1. The operator $L1_M f(x,y)$ interpolates each continuous function $f(x,y)$ in the points (X_{pq},Y_{pq}), $p,q=\overline{1,M}$, $p\neq q$:

$$L1_M f(X_{pq},Y_{pq})=f(X_{pq},Y_{pq}),\ p,q=\overline{1,M},\ p\neq q.$$

Proof.

By substituting the formula (1) by $x=X_{pq}$, $y=Y_{pq}$, we will get $L1_M f(X_{pq},Y_{pq})=\sum_{k=1}^{M}\sum_{\substack{l=1\\ l\neq k\\ l<k}}^{M} f(X_{kl},Y_{kl})h_{kl}(X_{pq},Y_{pq}).$

Consider that $h_{kl}(X_{pq},Y_{pq})=\prod_{\substack{j=1\\ j\neq k,l}}^{M}\frac{\omega_j(X_{pq},Y_{pq})}{\omega_j(X_{kl},Y_{kl})}$, and if $j=p$ or $j=q$, then $\omega_j(X_{pq},Y_{pq})=0$. That is, only when $k=p$, $l=q$ $h_{kl}(X_{pq},Y_{pq})=1$.

That is why

$$L1_M f(X_{pq},Y_{pq})=\sum_{k=1}^{M}\sum_{\substack{l=1\\ l\neq k\\ l<k}}^{M} f(X_{kl},Y_{kl})\Delta_{k,p}\Delta_{l,q}=f(X_{pq},Y_{pq}),$$

where $\Delta_{i,j}$ – is a Kronecker symbol.

So, $L1_M f(X_{pq},Y_{pq})=f(X_{pq},Y_{pq})$, $p,q=\overline{1,M}$, $p\neq q$.

Lemma 1 is proven.

Let's build the operator $L2_M f(x,y)$ as:

$$L2_M f(x,y)=\sum_{j=1}^{M}\int_{\Gamma_j^*} f(x,y)ds_j\,\frac{\prod_{\substack{k=1\\ k\neq j}}^{M}\omega_k(x,y)}{\int_{\Gamma_j^*}\prod_{\substack{k=1\\ k\neq j}}^{M}\omega_k(x,y)ds_j}. \qquad (2)$$

Lemma 2. The operator $L2_M f(x,y)$ has the following properties

$$L2_M f(X_{pq},Y_{pq})=0,\ p,q=\overline{1,M},\ p\neq q.$$

Proof.

Substituting the formula (2) by $x=X_{pq}$, $y=Y_{pq}$.

Consider that

$$L2_M f(X_{pq},Y_{pq})=\sum_{j=1}^{M}\int_{\Gamma_j^*} f(x,y)ds_j\,\frac{\prod_{\substack{k=1\\ k\neq j}}^{M}\omega_k(X_{pq},Y_{pq})}{\int_{\Gamma_j^*}\prod_{\substack{k=1\\ k\neq j}}^{M}\omega_k(x,y)ds_j},$$

because in some cases $k=p$ or $k=q$, we have $\omega_k(X_{pq},Y_{pq})=0$. That is $\prod_{\substack{k=1\\ k\neq j}}^{M}\omega_k(X_{pq},Y_{pq})=0$, $j=1,\dots,M$.

So, $L2_M f(X_{pq},Y_{pq})=0$, $p,q=\overline{1,M}$, $p\neq q$.

Lemma 2 is proven.

III. The Construction of the Operators of Interpolation of The Function of two Variables on a System of Intersections of Non-parallel Lines

Let's use the designation:

$$L_M f(x,y)=L1_M f(x,y)+L2_M(f(x,y)-L1_M f(x,y)) \qquad (3)$$

Theorem 1. The operator $L_M f(x,y)$ has the following properties:

1) $L_M f(X_{pq},Y_{pq})=f(X_{pq},Y_{pq}),\ p,q=\overline{1,M},\ p\neq q;$
2) $\int_{\Gamma_j^*} L_M f(x,y)ds_j=\int_{\Gamma_j^*} f(x,y)ds_j,\ j=\overline{1,M}.$

Proof.

1) From the Lemma 1 it follows that $L1_M f(X_{pq},Y_{pq})=f(X_{pq},Y_{pq})$. Therefore, considering that $L2_M(f(X_{pq},Y_{pq})-L1_M f(X_{pq},Y_{pq}))=0$ we will have

$$\begin{aligned}
&L_M f(X_{pq},Y_{pq})=\\
&=L1_M f(X_{pq},Y_{pq})+L2_M f(X_{pq},Y_{pq})-L2_M L1_M f(X_{pq},Y_{pq})=\\
&=L1_M f(X_{pq},Y_{pq})+L2_M f(X_{pq},Y_{pq})-\\
&\quad -L2_M L1_M f(X_{pq},Y_{pq})=\\
&\quad = f(X_{pq},Y_{pq})+L2_M f(X_{pq},Y_{pq})-L2_M f(X_{pq},Y_{pq})=\\
&\quad = f(X_{pq},Y_{pq}),\ p,q=\overline{1,M},\ p\neq q.
\end{aligned}$$

Thus, the first statement of Theorem 1 is proven.

2) To prove the statement 2 of the theorem, let's draw attention to the fact that when integrating the function $L2_M(f(x,y)-L1_M f(x,y))$ along the line Γ_j^* all the terms except the term when $k=j$ will be 0. That's why

$$\begin{aligned}
&\int_{\Gamma_j^*} L_M f(x,y)ds_j=\\
&=\int_{\Gamma_j^*}\left(L1_M f(x,y)+L2_M(f(x,y)-L1_M f(x,y))\right)ds_j=\\
&=\int_{\Gamma_j^*} L1_M f(x,y)ds_j+\\
&+\int_{\Gamma_j^*}\sum_{k=1}^{M}\int_{\Gamma_k^*}(f(x,y)-L1_M f(x,y))ds_k\,\frac{\prod_{\substack{i=1\\ i\neq k}}^{M}\omega_k(x,y)}{\int_{\Gamma_k^*}\prod_{\substack{i=1\\ i\neq k}}^{M}\omega_k(x,y)ds_k}ds_j=\\
&=\int_{\Gamma_j^*} L1_M f(x,y)ds_j+\\
&+\sum_{k=1}^{M}\int_{\Gamma_k^*}(f(x,y)-L1_M f(x,y))ds_k\,\frac{\int_{\Gamma_j^*}\prod_{\substack{i=1\\ i\neq k}}^{M}\omega_i(x,y)ds_j}{\int_{\Gamma_k^*}\prod_{\substack{i=1\\ i\neq k}}^{M}\omega_i(x,y)ds_k}=
\end{aligned}$$

$$= \int_{\Gamma_j^*} L1_M f(x,y)ds_j +$$

$$+\sum_{k=1}^{M} \int_{\Gamma_k^*} (f(x,y) - L1_M f(x,y))ds_k \frac{\delta_{k,j} \int_{\Gamma_j^*} \prod_{\substack{i=1 \\ i\neq k}}^{M} \omega_i(x,y)ds_j}{\int_{\Gamma_k^*} \prod_{\substack{i=1 \\ i\neq k}}^{M} \omega_i(x,y)ds_k} =$$

$$= \int_{\Gamma_j^*} L1_M f(x,y)ds_j +$$

$$+\int_{\Gamma_j^*} (f(x,y) - L1_M f(x,y))ds_j \frac{\int_{\Gamma_j^*} \prod_{\substack{i=1 \\ i\neq k}}^{M} \omega_i(x,y)ds_j}{\int_{\Gamma_j^*} \prod_{\substack{i=1 \\ i\neq k}}^{M} \omega_i(x,y)ds_j} =$$

$$= \int_{\Gamma_j^*} L1_M f(x,y)ds_j + \int_{\Gamma_j^*} f(x,y)ds_k - \int_{\Gamma_j^*} L1_M f(x,y))ds_j =$$

$$= \int_{\Gamma_j^*} f(x,y)ds_j,\ j = \overline{1,M}$$

Theorem 1 is proven.

Lemma 3. Every polynomial with $(M-1)$ degree with properties $P_{M-1}(X_{pq},Y_{pq}) = 0$ can be uniquely represented in the form $L2_M P_{M-1}(x,y) = \sum_{k=1}^{M} a_k \prod_{\substack{i=1 \\ i\neq k}}^{M} \omega_i(x,y)$.

Proof.

Since all the lines intersect with each other and there no intersection of more than two lines in one point, it is obvious that $L2_M P_{M-1}(X_{pq},Y_{pq}) = 0, p \neq q$ because each multiplication $\prod_{\substack{j=1 \\ j\neq k}}^{M} \omega_j(x,y)$ contains multiplier $\omega_p(x,y)$ or $\omega_q(x,y)$, which is equal to 0 in the point (X_{pq},Y_{pq}).

To prove that there are a_k, $k = \overline{1,M}$ not equal to 0, let's find them from the system

$$\int_{\Gamma_j^*} L2_M f(x,y)ds_j = \int_{\Gamma_j^*} f(x,y)ds_j = \gamma_j,\ j = 1,...,M$$

$$\int_{\Gamma_j^*} L2_M P_{M-1}(x,y)ds_j = \sum_{k=1}^{M} a_k \int_{\Gamma_j^*} \prod_{\substack{i=1 \\ i\neq k}}^{M} \omega_i(x,y)ds_j =$$

$$= a_j \int_{\Gamma_j^*} \prod_{\substack{i=1 \\ i\neq j}}^{M} \omega_i(x,y)ds_j = \gamma_j,$$

because for every $k \neq j$ the multiplication will include a multiplier $\omega_i(x,y) \neq 0$ on the line Γ_j.

Let's prove that $a_j = \dfrac{\gamma_j}{\int_{\Gamma_j^*} \prod_{\substack{i=1 \\ i\neq j}}^{M} \omega_i(x,y)ds_j}$, $j = \overline{1,M}$. This equality is obvious if $\int_{\Gamma_j^*} \prod_{\substack{i=1 \\ i\neq j}}^{M} \omega_i(x,y)ds_j \neq 0$.

We suppose that $f(x,y)$ is finite such that $\operatorname{supp} f(x,y) = D$ and each line Γ_k crosses this area so that the coordinates of the points $(X_{k_{\min}},Y_{k_{\min}})$, $(X_{k_{\max}},Y_{k_{\max}})$ satisfy the condition $\int_{\Gamma_j^*} \prod_{\substack{i=1 \\ i\neq j}}^{M} \omega_i(x,y)ds_j \neq 0$.

Remark. For the case $M = 3$ a proof that $\int_{\Gamma_j^*} \prod_{\substack{i=1 \\ i\neq j}}^{M} \omega_i(x,y)ds_j \neq 0$ is obvious if to apply an integration along the sides of the triangle.

Lemma 3 is proven.

Theorem 2. Each polynomial having $(M-1)$ degree can be uniquely represented by M projections along M lines Γ_j, $j = \overline{1,M}$, and by means of C_{M-1}^2 interpolation data in the points of intersection of these lines.

Proof.

The polynomial of M degree of two variables has $C_{M+2}^2 = \dfrac{(M+2)(M+1)}{2}$ coefficients. Polynomial of M-1 has $C_{M-1+2}^2 = C_{M+1}^2 = \dfrac{(M+1)M}{2}$ coefficients. Interpolation operator $L1f(x,y) = \sum_{0\le i+j\le M-2} f(X_{ij},Y_{ij})h_{ij}(x,y)$ has $C_{M-2+2}^2 = C_M^2 = \dfrac{M(M-1)}{2}$ coefficients.

Then the formula for the polynomial $L_M f(x,y) = L1_M f(x,y) + L2_M(f(x,y) - L1_M f(x,y))$ will have the degree $M-1$.

Polynomial $L1_M f(x,y)$ has $(M-2)$ degree with properties $L1_M f(x,y) \equiv f(x,y),\ \forall f(x,y) \in P_{M-2}$, where $P_{M-2} = \left\{ \sum_{0\le i+j\le M-2} a_{ij}x^i y^j,\ a_{ij} \in \mathbb{R} \right\}$ – a set of polynomials of two variables of degree $(M-2)$. Let's build the operator $L1_M P_{M-2}(x,y) = \sum_{r=1}^{M-1} \sum_{s=r+1}^{M} P_{M-2}(X_{rs},Y_{rs})h_{rs}(x,y)$, where $h_{rs}(x,y) = \prod_{\substack{j=1 \\ j\neq r,s}}^{M} \dfrac{\omega_j(x,y)}{\omega_j(X_{rs},Y_{rs})}$. These functions are the polynomials of degree $(M-2)$, because each function $h_{rs}(x,y)$ is the basic polynomial of the Lagrangian interpolation in $C_M^2 = \dfrac{M(M-1)}{2}$ intersection points of M lines, having no more than one intersection at one point. As, $C_M^2 + M = \dfrac{M(M-1)}{2} + M = M\left(\dfrac{M-1+2M}{2}\right) = \dfrac{M(M+1)}{2}$, then the system of functions $h_{rs}(x,y)$ $r,s = \overline{1,M}$, $r \neq s$ and $\dfrac{\prod_{\substack{r=1 \\ r\neq s}}^{M} \omega_r(x,y)}{\int_{\Gamma_s^*} \prod_{\substack{r=1 \\ r\neq s}}^{M} \omega_r(x,y)ds_s}$ create a complete linearly independent system of polynomials with degree $\le M-1$. Let us consider the function $L_M^* f(x,y) = L2_M(f(x,y) - L1_M f(x,y))$, which has the properties:

1) This function is a polynomial of degree $M-1$;
2) $L2_M^* P_{M-2}(X_{rs},Y_{rs}) = 0$, $r,s = \overline{1,M}$, $r \neq s$.

According to the assumption of the lemma 3, the polynomial $L2_M^* f(x,y)$ can be found by use of M projections along the specified system of straight lines. It can be represented as

$$L2_M^* f(x,y) =$$

$$= \sum_{j=1}^{M} \int_{\Gamma_j^*} (f(x,y) - L1_M f(x,y)) ds_j \frac{\prod_{\substack{k=1 \\ k\neq j}}^{M} \omega_k(x,y)}{\int_{\Gamma_j^*} \prod_{\substack{k=1 \\ k\neq j}}^{M} \omega_k(x,y) ds_j}$$

because of assumption $\int_{\Gamma_j^*} \prod_{\substack{k=1 \\ k\neq j}}^{M} \omega_k(x,y) ds_j \neq 0$.

From here we will receive

$$L_M f(x,y) = L1_M f(x,y) + L2_M (f(x,y) - L1_M f(x,y)$$

which satisfies all the necessary properties of the theorer

Theorem 2 is proven.

IV. An Integral Representation of the Remainde the Approximation

Theorem 3. If $f(x,y)$ belongs to the class $\mathbb{C}^M($ $D = \operatorname{supp} f(x,y)$, then the error of approxima $Rf(x,y) = f(x,y) - L_M f(x,y)$ can be presented in integral form

$$Rf(x,y) = r_{M-1} f(x,y) - L_M (r_{M-1} f(x,y)).$$

Proof. The M times differentiated function can be represented as a Taylor formula

$$f(x,y) = T_{M-1} f(x,y) +$$

$$+\int_0^1 \frac{\partial^M}{\partial t^M} f(X_{pq} + t(x - X_{pq}), Y_{pq} + t(y - Y_{pq})) \cdot \frac{(1-t)^{M-1}}{(M-1)!} dt,$$ where $T_{M-1} f(x,y) = \sum_{0 \le r+s \le M-1} f^{(r,s)}(X_{pq}, Y_{pq}) \frac{(x - X_{pq})^r (y - Y_{pq})^s}{r!s!}$ – is Taylor's polynomial with a degree $(M-1)$ of decomposition of functions $f(x,y)$ in the vicinity of the point (X_{pq}, Y_{pq}) .Let's take into account that $T_{M-1} f(x,y)$ is Taylor's polynomial of the function $f(x,y)$ with a degree $(M-1)$, and operators $L_M f(x,y)$ restore precisely all the polynomials with a degree $(M-1)$. According to the lemma, we can write the following set of equations:

$$Rf(x,y) = f(x,y) - L_M f(x,y) =$$
$$= T_{M-1} f(x,y) + r_{M-1} f(x,y) - L_M [T_{M-1} f(x,y) + r_{M-1} f(x,y)] =$$
$$= T_{M-1} f(x,y) + r_{M-1} f(x,y) -$$
$$-[L_M T_{M-1} f(x,y) + L_M r_{M-1} f(x,y)] =$$
$$= T_{M-1} f(x,y) + r_{M-1} f(x,y) - L_M T_{M-1} f(x,y) - L_M r_{M-1} f(x,y) =$$
$$= T_{M-1} f(x,y) + r_{M-1} f(x,y) - T_{M-1} f(x,y) - L_M r_{M-1} f(x,y) =$$
$$= r_{M-1} f(x,y) - L_M r_{M-1} f(x,y).$$

Theorem 3 is proven.

Theorem 4. In the case when interpolation data are unknown $f(X_{rs}, Y_{rs})(\omega_r(X_{rs}, Y_{rs}) = 0, \omega_s(X_{rs}, Y_{rs}) = 0;$ $r, s \in \{1, 2, ..., M\}, r \neq s$

V. Example

Let us consider a triangle with vertices $A_1(R,0), A_2\left(-\frac{R}{2}, \frac{\sqrt{3}R}{2}\right), A_3\left(-\frac{R}{2}, -\frac{\sqrt{3}R}{2}\right)$ and equations of the sides (see Fig. 1):

$$\Gamma_{12} : \omega_{12}(x,y) = (x - X_1)(Y_2 - Y_1) - (y - Y_1)(X_2 - X_1) = 0$$

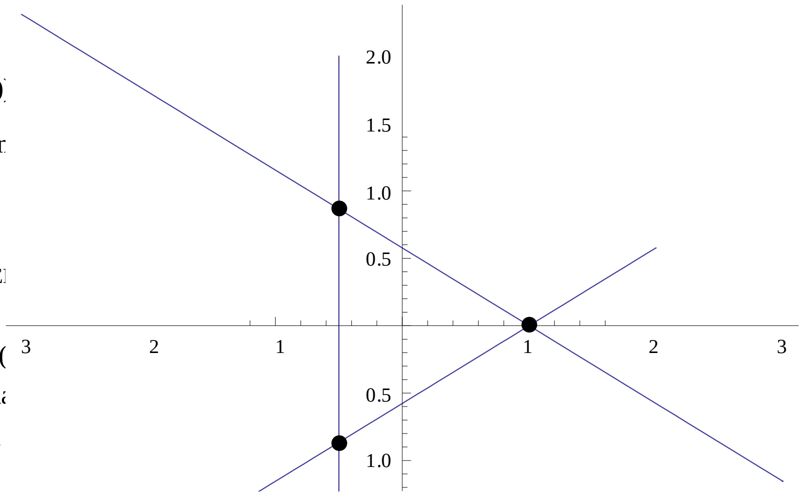

Fig. 1. Case study: the system of lines.

Let's build the operator $L1 f(x,y)$, which interpolates $f(x,y)$ at the vertices of the triangle $(X_k, Y_k), k = \overline{1,3}$ as:

$$L1 f(x,y) = f(X_1, Y_1) \cdot \frac{\omega_{23}(x,y)}{\omega_{23}(X_1, Y_1)} +$$
$$+ f(X_2, Y_2) \cdot \frac{\omega_{13}(x,y)}{\omega_{13}(X_2, Y_2)} + f(X_3, Y_3) \cdot \frac{\omega_{12}(x,y)}{\omega_{12}(X_3, Y_3)}$$

$$L2 f(x,y) = \frac{\omega_{23}(x,y)\omega_{31}(x,y)}{\int_{\Gamma_{12}} \omega_{23}(x,y)\omega_{31}(x,y) ds_{12}} \cdot \int_{\Gamma_{12}} f(x,y) ds_{12} +$$
$$+ \frac{\omega_{12}(x,y)\omega_{31}(x,y)}{\int_{\Gamma_{23}} \omega_{12}(x,y)\omega_{31}(x,y) ds_{23}} \cdot \int_{\Gamma_{23}} f(x,y) ds_{23} +$$
$$+ \frac{\omega_{12}(x,y)\omega_{23}(x,y)}{\int_{\Gamma_{31}} \omega_{12}(x,y)\omega_{23}(x,y) ds_{31}} \cdot \int_{\Gamma_{31}} f(x,y) ds_{31}$$

Let's use the designation

$$Lf(x,y) = L1 f(x,y) + L2(f(x,y) - L1 f(x,y)).$$

Let's check the properties of the operator $Lf(x,y)$:

1) $Lf(X_i, Y_i) = f(X_i, Y_i), i = \overline{1,3},$

2) $\int_{\Gamma_{ij}} Lf(x,y) ds_{ij} = \int_{\Gamma_{ij}} f(x,y) ds_{ij}.$

1) By substituting the formula $Lf(x,y)$ by the point $(X_i, Y_i), i = \overline{1,3}$ we'll have

$$Lf(X_i,Y_i) = L1f(X_i,Y_i) + L2(f(X_i,Y_i) - L1f(X_i,Y_i)) =$$
$$= L1f(X_i,Y_i) + L2f(X_i,Y_i) - L2L1f(X_i,Y_i) =$$
$$= f(X_i,Y_i) + L2f(X_i,Y_i) - L2f(X_i,Y_i) = f(X_i,Y_i), i = \overline{1,3}.$$

2) Let us consider the integral

$$\int_{\Gamma_{ij}} Lf(x,y)ds_{ij} = \int_{\Gamma_{ij}} \left(L1f(x,y) + L2(f(x,y) - L1f(x,y))\right)ds_{ij} =$$
$$= \int_{\Gamma_{ij}} L1f(x,y)ds_{ij} + \int_{\Gamma_{ij}} L2(f(x,y) - L1f(x,y))ds_{ij} =$$
$$= \int_{\Gamma_{ij}} L1f(x,y)ds_{ij} + \int_{\Gamma_{ij}} L2f(x,y)ds_{ij} - \int_{\Gamma_{ij}} L2L1f(x,y)ds_{ij} =$$
$$= \int_{\Gamma_{ij}} L1f(x,y)ds_{ij} + \int_{\Gamma_{ij}} L2f(x,y)ds_{ij} - \int_{\Gamma_{ij}} L1f(x,y)ds_{ij} =$$
$$= \int_{\Gamma_{ij}} L2f(x,y)ds_{ij} = \int_{\Gamma_{ij}} f(x,y)ds_{ij},$$
$$(i,j) = \{(1,2),(2,3),(3,1)\}.$$

To verify that $Lf(x,y) \equiv f(x,y)$ for all $f(x,y) \in P_2$.

Let's build the operator $L1f(x,y)$ as:

$$L1P_2(x,y) = P_2(X_1,Y_1)\frac{\omega_{23}(x,y)}{\omega_{23}(X_1,Y_1)} + P_2(X_2,Y_2)\frac{\omega_{31}(x,y)}{\omega_{31}(X_2,Y_2)} +$$
$$+P_2(X_3,Y_3)\frac{\omega_{12}(x,y)}{\omega_{12}(X_3,Y_3)}$$

This polynomial has the properties

$$L1P_2(X_k,Y_k) = P_2(X_k,Y_k), k = \overline{1,3}$$

Polynomial $P_2^*(x,y) = P_2(x,y) - L1P_2(x,y)$ is a polynomial of the second degree with properties $P_2^*(X_k,Y_k) = 0, k = \overline{1,3}$. According to the lemma 3, every such 2nd-degree polynomial can be unambiguously represented by the formula

$$P_2(x,y) = \frac{\omega_{13}(x,y)\omega_{12}(x,y)}{\int_{\Gamma_{23}} \omega_{13}(x,y)\omega_{12}(x,y)ds_{23}} \cdot \gamma_{23} +$$
$$+\frac{\omega_{12}(x,y)\omega_{23}(x,y)}{\int_{\Gamma_{13}} \omega_{12}(x,y)\omega_{23}(x,y)ds_{13}} \cdot \gamma_{13} + \frac{\omega_{23}(x,y)\omega_{13}(x,y)}{\int_{\Gamma_{12}} \omega_{23}(x,y)\omega_{13}(x,y)ds_{12}} \cdot \gamma_{12}$$

Thus, the operator of the function's interpolation $f(x,y)$ in the vertices of the triangle $(X_k,Y_k), k = \overline{1,3}$ with given projections along each side of the triangle can be represented as:

$$L1f(x,y) = f(X_1,Y_1)\frac{\omega_{23}(x,y)}{\omega_{23}(X_1,Y_1)} + f(X_2,Y_2)\frac{\omega_{13}(x,y)}{\omega_{13}(X_2,Y_2)} +$$
$$+f(X_3,Y_3)\frac{\omega_{12}(x,y)}{\omega_{12}(X_3,Y_3)}$$

$$Lf(x,y) = L1f(x,y) + L2(f(x,y) - L1(x,y)).$$

So, $Lf(x,y) \equiv f(x,y)$ for all $f(x,y) \in P_2$.

VI. Conclusion

The paper proposes a new method for construction of interpolation operators for the functions of 2 variables by means of the known projections - integrals along a given system of M lines. The method supposes, that there is no more than one intersection of the lines at one point. Unknown interpolation data are found from the condition that the approximation operator accurately restores the polynomials of degree $(M-1)$. All results are formulated and proven in the form of lemmas and theorems. The theorem is proven that each arbitrary polynomial of $(M-1)$ degree of two variables can be uniquely represented by means of only M projections along a given system of lines if all other unknown coefficients $(C_{M+2}^2 - M)$ are selected from the condition that the obtained operator $L_M f(x,y)$ accurately restores all the polynomials of $(M-1)$ degree. An integral representation of the remainder of approximation of the constructed operators for M times continuously differentiable function of 2 variables is found.

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Vitaliy Mezhuyev received a degree in informatics from Berdyansk State Pedagogical University (BSPU), Ukraine, in 1997. In 2002, he received a PhD in educational technology from Kiev National Pedagogical University and, in 2012, an ScD in information technology from Odessa National Technical University, Ukraine. From 2004 until 2014, he was a head of the Department of Informatics and Software Engineering at BSPU, Ukraine. Now he is a professor at Faculty of Computer Systems and Software Engineering in University Malaysia Pahang, head of the Software Engineering Research Group. During his career, Vitaliy Mezhuyev participated in multiple international scientific and industrial projects, devoted to the formal modelling, design, and development of advanced software systems as a network-centric real-time operating system; IDEs for the automation of development of parallel real-time applications; tools for specification, verification and validation of software products; visual environment for metamaterials modelling and others. His current research interests include formal methods, metamodeling, safety modelling and verification of hybrid software systems, and the design of cyber-physical systems.

Oleh M. Lytvyn received a specialist degree in physics and mathematics from Berdyansk State Pedagogical University (BSPU), Ukraine, in 1964. In 1968, he received a PhD in computational methods from A. Pidgorny Institute of Mechanical Engineering Problems (Kharkiv) and in 1989, a ScD in mathematical modelling and computational methods from Shevchenko National University, Ukraine. From 1972 to 1980, he was a senior researcher in the laboratory of mathematical modelling of the Research Institute VODGEO. From 1980 and till now he works at the Ukrainian Engineering-Pedagogical Academy (UEPA). From 1995 to 2009, he was a head of the Department of Applied Mathematics, and from 2009, he was a head of the Department of Higher and Applied Mathematics at the UEPA. All this time Prof. Oleh Lytvyn carried out a research work, resulted in the creation of new scientific fields: the theory of interlination, interflatation, interstripation of functions of many variables; new methods of solving problems of computer tomography (flat and spatial); the theory of the approximation of differential operators; the theory of creating optimal cubature formulas; optimal finite element methodologies; the theory of dividiral and multigral calculations; the method of linear integral- differential equations, the methods of nonlinear integral- differential equations, the method of finite elements; the theory of operators that give a rise to the vector-based problem of seismic tomography using data from seismic probing of a planet. In 2009, O.N. Lytvyn had become the laureate of the State Prize of Ukraine in the field of science and technology.

Oleg O. Lytvyn received a specialist degree in computational methods from Kharkiv National University of Radio Electronics, Ukraine, in 1992. In 2010, he received a PhD in mathematical modeling and computational methods from the Institute of Cybernetics of National Academy of Science of Ukraine (IC of NAS) and, in 2017, a ScD in mathematical modeling and computational methods from the IC of NAS. From 2000 and till now he works at the Ukrainian engineering-pedagogical academy (UEPA). From 2013 he is a dean of the Faculty of Technology of UEPA. His research is devoted to the theory of operators and vector-based problems of seismic tomography using data from the seismic probing of a planet.

Iuliia Pershyna received a specialist degree in mathematics and informatics from Kharkiv National University, Ukraine, in 2000. In 2007, she received a PhD in mathematical modeling and computational methods from Kharkiv National University of Radio Electronics and, in 2015, an ScD in mathematical modeling and computational methods from Institute of Cybernetics of National Academy of Science of Ukraine, Kiev. From 2000, she works at the Ukrainian Engineering-Pedagogical Academy. Her research is devoted to developing the theory of approximation of discontinuous functions by discontinuous splines and its application to solve the two-dimensional problem of computed tomography and 3D and 4D mathematical modelling in computer tomography.

Olesia Nechuiviter received a specialist degree in mathematics and informatics from Kharkiv National University, Ukraine, in 1994. In 2006, she received a PhD in mathematical modeling and computational methods from Institute of Cybernetics of National Academy of Science of Ukraine (IC of NAS) and, in 2015, an ScD in mathematical modeling and computational methods from IC of NAS. During 1994 to 2007, she was working at the Department of Applied Mathematics in Ukrainian Engineering and Pedagogical Academy, Kharkiv. Since 2017 she is a head of Department of Information Computer and Polygraphy Technologies. Her researches are devoted to the unification of the results in the theory of optimal algorithms, the theory of interlineation and interflatation (the blending-function interpolation) of many variables functions and the Fourier's analysis.

Khurdei Evheniia received a specialist degree in mathematics and informatics from Kharkiv National University, Ukraine, in 2011. From 2011 to 2014, she was a PhD student in speciality Mathematical modelling and computational methods of Ukrainian Engineering Pedagogical Academy. From 2014, she works at the Ukrainian engineering-pedagogical academy. Her research interests are development of mathematical models of the internal structure of bodies by the methods of computed tomography using interlineation and interflatation of functions.