Distribution of Number of Edges on Surface of a Sphere

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***Abstract*—In this article we consider N spherical caps of area $4\pi p$ were uniformly distributed over the surface of a unit sphere. We study the random intersection graph G_N constructed by these caps. We prove that for $p = \frac{c}{N^{\alpha}}, c > 0$ and $\alpha > 2$, the number of edges in graph G_N follow the Poisson distribution.**

***Index Terms*—Connectivity distance, edges, spherical caps, wireless LAN, wired LAN.**

I. Introduction

Sensor Network poses a number of challenging problems such as converge, connectivity and tracking. We are considering a typical network consisting two major physical components. First, we have "*station*". A station is an endpoint of connection with a wireless interface used to access the network. Typical examples of stations are laptop, mobile etc.

Second the "*access point*". An access point has one wireless interface and one wired interface. The wired interface is effective between different access points and the wireless interface is effective between the stations and the access point. It is therefore the access point that connects the wireless LAN to the wired LAN. Also the radio propagation effects limit the range of wireless transmission. This range can be increased by increasing the transmission power.

We deployed some access point on the surface of unit sphere. The transmission range of a access point draw a "*spherical cap*" on the surface of unit sphere and every station with in this spherical cap can communicate with the access point. Also we consider that the station is free to move on the surface of this unit sphere. If the two spherical caps intersect, it provides a covered path by which a mobile station can move from one access point to the other access point without disconnecting from the network.

Our model can be representing as a random intersection graph. Random intersection graphs were introduce in [8], and defined as:

Let us consider a set V with n vertices and another set of objects W with m objects. Define a bipartite graph $G^*(n,m,p)$ with independent vertex sets V and W. Edges between $v \in V$ and $w \in W$ exists independently with probability p. The random intersection graph $G(n,m,p)$ derived from $G^*(n,m,p)$ is defined on the vertex set V with vertices $v_1, v_2 \in V$ are adjacent if and only if there exists some $w \in W$ such that both v_1 and v_2 are adjacent to w in $G^*(n,m,p)$.

In our model we consider the set of access point as vertex sets V and W are the set of points in the transmission range of different access points on the surface of unit sphere. The cover path between to access points is considered as an edge. Also we define W_v be a random subset of W such that each element of W_v is adjacent to $v \in V$. Any two vertices $v_1, v_2 \in V$ are adjacent if and only if $W_{v_1} \cap W_{v_2} \neq \phi$, and edge set $E(G)$ is define as:

$$E(G) = \left\{ \left(v_i, v_j\right) : v_i, v_j \in V,\ W_{v_i} \cap W_{v_j} \neq \phi \right\}.$$

Dudley [5], derived the distribution of the degree of a vertex of random intersection graph. Also show that if n be the number of vertices and $\lfloor n^{\alpha} \rfloor$ be the number of objects, the vertex degree changes sharply between $\alpha < 1, \alpha = 1$ and $\alpha > 1$. Bhupendra Gupta [3], derived the strong threshold for the connectivity between any two arbitrary vertices of vertex set V, and determines the almost sure probability bounds for the vertex degree of a typical vertex of random intersection graph.

II. Our Model

In this paper we considered the random intersection graph generated by the spherical caps on the surface of a 3-dimensional unite sphere.

Let $C_1, C_2, ..., C_N$ be the spherical caps and $X_1, X_2, ..., X_N$ are their respective centers on the surface of a unit sphere. Let $X_1, X_2, ..., X_N$ are uniformly distributed over the surface of unit sphere. Now define a random intersection graph G_N on the surface of unite sphere, with vertex set $\chi_N = \left\{X_1, X_2, ..., X_N\right\}$ and edge set $\varepsilon_N = \left\{X_i X_j : C_i \cap C_j \neq \phi, i \neq j\right\}$.

The aim of this paper is to investigate the evolution of edges in the graph G_N with vertex set $\chi_N = \left\{X_1, X_2, ..., X_N\right\}, N = 1, 2, ...$ where the vertices are independently and uniformly distributed on the surface of a unit sphere. H. Maehara [6] gives the asymptotic results for the various properties of random intersection graph of random spherical caps on surface of unit sphere. Also Bhupendra Gupta [2] gives the strong threshold function

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$p_0(N)=o\left(\frac{\log N}{N}\right)$ for the coverage of the surface of a unit sphere by the spherical caps. Bhupendra Gupta has shown that for large N, if $\frac{N_p}{\log N}>\frac{1}{2}$ the surface of sphere is completely covered by N caps almost surely, and if $\frac{N_p}{\log N}\le\frac{1}{2}$ a partition of the surface of sphere is remains uncovered by N caps almost surely.

III. Supporting Results

Let $C_1, C_2, ..., C_N$ be the spherical caps on the surface of a unit sphere with their centers $X_1, X_2, ..., X_N$ and uniformly distributed over the surface of unit sphere. We defined a random intersection graph G_N on the surface of unite sphere, with vertex set $\chi_N=\left\{X_1, X_2, ..., X_N\right\}$ and edge set $\varepsilon_N=\left\{X_iX_j : C_i\cap C_j\neq\phi, i\neq j\right\}$.

Let $p=p(a)$ be the probability at a point $'x'$ on the surface of unit sphere is covered by a specified spherical cap of angular radius $'a'$. Then the area of the spherical cap of angular radius $'a'$ is equal to $4\pi p$.

Poisson Approximation:

Let $|\varepsilon|$ denote the cardinality of the edge set i.e., the number of edges in the graph G_N. Define an indicator function

$$\varepsilon_i=\begin{cases}1, & C_i\cap C_j\neq\phi, i\neq j;\\ 0, & \text{otherwise}\end{cases}, \tag{1}$$

i.e., if X_i is an end point of an edge, then $\xi_i=1$ and hence $|\varepsilon|=\sum_{i\in I}\xi_i$, where $I=\left\{i : X_iX_j\in\varepsilon, i\neq j\right\}$ is the index set.

$$\begin{aligned}E|\varepsilon| &= E\left[\sum_{i=1}^{n}\xi_i\right]=\sum_{i=1}^{n}E\left[\xi_i\right]=\binom{N}{2}4p(1-p)\\ &= 2N(N-1)p(1-p)\le 2N^2p(1-p).\end{aligned} \tag{2}$$

The total variation distance between two integer valued random variables X and Y is define as

$$d_{TV}(X,Y)=\sup_{u\in\Omega}\left|F_X(u)-F_Y(u)\right|$$

The following theorem gives upper bound of d_{TV} of binomial and Poisson random variables.

Theorem 1: (Arratia 1989 [1]): Suppose $\xi_i, i\in I$ is a finite collection of Bernoulli random variables. Set $p_i=E\left[\xi_i\right]=P\left[\xi_i=1\right]$, and $p_{ij}=E\left[\xi_i\xi_j\right]$. Let $\lambda=\sum_{i\in I}p_i$, and suppose λ is finite. Let $|\varepsilon|=\sum_{i\in I}\xi_i$. Then

$$\begin{aligned}&d_{TV}\left(|\varepsilon|, Po(\lambda)\right)\\ &\le\min(3,\lambda^{-1})\left(\sum_{i\in I}\sum_{j\in\mathcal{N}_i\setminus\{i\}}p_{ij}+\sum_{i\in I}\sum_{j\in\mathcal{N}_i}p_ip_j\right).\end{aligned}$$

where $\mathcal{N}_i$ is the adjacency neighborhood of i, i.e. the set $\{i\}\cup\left\{j\in I : X_iX_j\in\varepsilon\right\}$,

where $Po(\lambda)$ denote Poisson random variable with parameter λ.

IV. Main Results

The following theorem gives the distribution of number of edges of the surface of a unit sphere.

Theorem 2: For $p=p(a)=\frac{c}{N^{\alpha}}$, where $c>0$ and $\alpha>2$. Then for sufficiently large N,

$$d_{TV}\left(|\varepsilon|, Po(\lambda)\right)\to 0, \tag{3}$$

i.e., the number of edges in graph G_N is a Poisson random variable with parameter $\lambda=\sum_{i\in I}p_i<\infty$.

Proof: First we consider

$$p_i=E\left[\xi_i\right]=P\left[\xi_i=1\right]. \tag{4}$$

We know there exists an edge between X_i and X_j if and only if $C_i\cap C_j\neq\phi$, i.e. the distance between X_i and X_j is less than $2a$. Now consider another spherical cap D_i centered at X_i and of radius $2a$.

$$\begin{aligned}P\left[\xi_i=1\right] &= P\left[C_i\cap C_j\neq\phi\right]\\ &= P\left[X_j\in D_i\right]=p(2a).\end{aligned} \tag{5}$$

Now by using the equation (2.1) of [2] we have

$$p=p(a)=\sin^2\left(a/2\right). \tag{6}$$

Using (6) in (5), we get

$$\begin{aligned}P\left[\xi_i=1\right] &= \sin^2(a)=\frac{1}{2}\left(1\text{-}\cos(2\mathrm{a})\right)\\ &= 4p(1-p).\end{aligned} \tag{7}$$

Using (7) in (4), we get

$$p_i=E\left[\xi_i\right]=4p(1-p). \tag{8}$$

Now consider

$$
\begin{aligned}
p_{ij} &= E\left[\xi_i \xi_j\right] = P\left[\xi_i \xi_j = 1\right] \\
&= \sum_{l=1, l\neq i}^{n} \sum_{k=1, k\neq j}^{n} P\left[\left(C_i \cap C_l\right) \neq \phi, \left(C_k \cap C_j\right) \neq \phi\right] \\
&\quad - P\left[\left(C_i \cap C_j\right) \neq \phi\right] \\
&= \sum_{l=1, l\neq i}^{n} P\left[\left(C_i \cap C_l\right) \neq \phi\right] \sum_{k=1, k\neq j}^{n} P\left[\left(C_k \cap C_j\right) \neq \phi\right] \\
&\quad - P\left[\left(C_i \cap C_j\right) \neq \phi\right] \qquad (9) \\
&= \left(4(N-1)p(1-p)\right)^2 - 4p(1-p) \\
&= 16\left((N-1)p(1-p)\right)^2 \left(1 - \frac{1}{4(N-1)^2 p(1-p)}\right) \\
&\leq 16\left((N-1)p(1-p)\right)^2 .
\end{aligned}
$$

Now by Theorem 1, we have

$$
\begin{aligned}
& d_{TV}\left(\left|\varepsilon\right|, Po(\lambda)\right) \\
& \leq \min(3, \lambda^{-1}) \left(\sum_{i\in I} \sum_{j \in \mathcal{N}_i \setminus \{i\}} p_{ij} + \sum_{i \in I} \sum_{j \in \mathcal{N}_i} p_i p_j \right).
\end{aligned}
$$

Using eqs. (8) and (9), we get

$$
\begin{aligned}
& d_{TV}\left(\left|\varepsilon\right|, Po(\lambda)\right) \\
& \leq \min(3, \lambda^{-1}) \begin{pmatrix} \sum_{i\in I} \sum_{j \in \mathcal{N}_i \setminus \{i\}} \left(4(N-1)p(1-p)\right)^2 \\ + \sum_{i\in I} \sum_{j \in \mathcal{N}_i} 4p(1-p) 4p(1-p) \end{pmatrix} \\
& \leq \min(3, \lambda^{-1}) \begin{pmatrix} \frac{N(N-1)^3}{2} \left(4p(1-p)\right)^2 \\ + \frac{N(N-1)}{2} \left(4p(1-p)\right)^2 \end{pmatrix}.
\end{aligned}
$$

Taking $p = \frac{c}{N^{\alpha}}$ and $\alpha > 2$ in above, we get

$$
d_{TV}\left(\left|\varepsilon\right|, Po(\lambda)\right) \to 0, \qquad \text{as } N \to \infty.
$$

V. Conclusion

Theorem 2 shows that the total variation Difference between ε and $Po(\lambda)$ converges to 0. This implies that the number of edge on the surface of unit sphere follows Poisson distribution with parameter $\lambda = \sum_{i \in I} p_i$.

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